ON THE MULTIPLICATIVE PRODUCTS OF THE
n-DIMENSIONAL DISTRIBUTIONAL HANKEL TRANSFORMS

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1. Abstract. In this note, we prove several multiplicative products of the
n-dimensional Hankel transform. In fact the following formulae:

\[ \mathcal{H}\{\delta^{(k)}(u(x)\}\mathcal{H}\{\delta^{(\ell)}(u(x))\} = D\mathcal{H}\{\delta^{(k+\ell+n-2)}(u(x))\}, \quad (\text{III,1}) \]

\[ \mathcal{H}\{\delta^{(k)}(P)\mathcal{H}\{\delta^{(\ell)}(P)\} = C\mathcal{H}\{\delta^{(k+\ell+n-2)}(P)\}, \quad (\text{IV,9}) \]

\[ \mathcal{H}\{\delta^{(k)}(|x|^2)\mathcal{H}\{\delta^{(\ell)}(|x|^2)\} = C\mathcal{H}\{\delta^{(k+\ell+n-2)}(|x|^2)\}, \quad (\text{V,1}) \]

We observe that the above results were inspired in a non-edited paper due to Manuel
Aguirre Téllez (cfr. [8]).

II. Introduction.

We begin with some definitions. Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of the n-
dimensional Euclidean space \( \mathbb{R}^n \). Consider a non-degenerate quadratic form in n
variables of the form

\[ P = P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \quad (\text{II,1}) \]

where \( n = p + q \).

We define the two following distributions, as follows

\[ P^\lambda_+ = \begin{cases} 
    P^\lambda & \text{if } P > 0, \\
    0 & \text{if } P \leq 0;
\end{cases} \quad (\text{II,2}) \]
and
\[ P^\lambda = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^\lambda & \text{if } P \leq 0. \end{cases} \]  

(II, 3)

\( \mathcal{H} \) denotes the distributional Hankel transform. Let \( \phi(t) \) be defined in \( \mathbb{IR}^+: \{t, t > 0\} \).

By the Hankel transform of the function \( \phi(t) \) we mean the function \( g(s), 0 \leq s < \infty \), defined by the formula
\[ g(s) = \mathcal{H}\{\phi(t)\} = \int_0^\infty \phi(t) J_\nu(xt) \sqrt{x} t \, dt, \]  

(II, 4)

or, equivalently,
\[ g(s) = (\mathcal{H}\{\phi(t)\}) = \frac{1}{2} \int_0^\infty \phi(t) t^{\frac{n-2}{2}} R_{\frac{n-2}{2}}(\sqrt{st}) \, dt, \]  

(II, 5)

\[ R_m(x) = \frac{J_m(x)}{x^m}, \]  

(II, 6)

and \( J_m(x) \) is the well-known Bessel function defined by the formula
\[ J_m(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left( \frac{x}{2} \right)^{m+2\nu}}{\nu! \Gamma(m + \nu + 1)}. \]  

(II, 7)

It is well known (cfr. [1], p. 240) that if \( \phi(t) \) satisfies adequate conditions, for example if \( \phi(t) \) belongs to \( S_{\mathbb{IR}^+} \), the following formula is valid:
\[ \phi(t) = (\mathcal{H}\{g(s)\}) = \frac{1}{2} \int_0^\infty g(s) s^{\frac{n-2}{2}} R_{\frac{n-2}{2}}(\sqrt{st}) \, ds. \]  

(II, 8)

Let \( S_{\mathbb{IR}^+} \) designate the space of functions \( f \in S \) defined in the positive half line \( \mathbb{IR}^+: \{t, t > 0\} \). By \( S'_{\mathbb{IR}^+} \) we designate the dual of \( S_{\mathbb{IR}^+} \).

Let \( U(t) \in S'_{\mathbb{IR}^+} \). The Hankel transform of \( U(t) \) will be, by definition, the distribution \( \nu(s) \in S'_{\mathbb{IR}^+} \), defined by the formula
\[ \langle \mathcal{H}\{U(t), \phi(s)\} \rangle = \langle U(t), (\mathcal{H}\{\phi(s)\}) \rangle, \]  

(II, 9)

for every \( \phi \in S_{\mathbb{IR}^+} \).

There are other definitions of the Hankel transform of distributions (cfr. [2]) but, here we use the definition which appears in [3], Appendix I, p. 64, especially, Theorem 26, p. 72. In fact, we have that
\[ \mathcal{H}^{-1}(\tilde{T}) = \{T\}^\wedge, \]  

(II, 10)
here $\tilde{T}$ is the image of $T$ belongs to $S'_{R^n}$ in $S'$, defined by the formula
\[ \langle \tilde{T}, \phi(t) \rangle = \langle T, \phi(r^2) \rangle , \]  
(II, 11)
for every $\phi \in S_{R^+}$.

We designate $S_{R^n}^\ast$ the family of functions $f(x)$ belongs to $S_{R^n}$ and, further, invariable by rotations. Moreover, $S_{R^n}^\ast$ designates the dual of $S_{R^n}^\ast$.

Following strictly the definitions of [8], we shall define the $k$-th derivative of Dirac delta in $u(x_1, x_2, \ldots, x_n)$.

Let $\phi_t$ denote a distribution of one variable $t$. Let $u \in C^\infty(\mathbb{R}^n)$ be such that $(n - 1)$-dimensional manifold $u(x_1, x_2, \ldots, x_n) = 0$ has no critical point. By $\phi_{u(x)}$ (cfr. [9], page 102) we designate the distribution defined on $\mathbb{R}^n$ by
\[ \langle \phi_{u(x)}, \varphi(x) \rangle = \langle \phi_t, \psi(t) \rangle , \]  
(II, 12)
where
\[ \psi(t) = \int_{u(x)=t} \varphi(x)w_u(x, dx) , \]  
(II, 13)
and $\varphi \in C^\infty_0(\mathbb{R}^n)$ is the set of infinitely differentiable functions with compact support and $w_u$ is a $(n - 1)$-dimensional exterior differential form on $u$ defined as follows:
\[ du \wedge dw = dx_1 \wedge \ldots \wedge dx_n , \]  
(II, 14)
and the orientation of the manifold $u(x) = t$ is such that $w_u(x, dx) > 0$.

On the other hand (cfr. [10], p. 230, form. (6)), we have
\[ (\delta^{(k)}(G(x_1, \ldots, x_n), \varphi(x_1, \ldots, x_n)) = (-1)^k \int_{G(x)=0} w_k(\varphi) , \]  
(II, 15)
k = 0, 1, \ldots; where $x = (x_1, \ldots, x_n)$, $G(x_1, \ldots, x_n)$ is such an infinitely differentiable function that
\[ \text{grad } G = \left( \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n} \right) \neq 0 , \]  
(II, 16)
\[ w_k(\varphi) = \frac{\partial^k}{\partial u_1^k} \left\{ D \left( \frac{x}{u} \right) \varphi, (u_1, \ldots, u_n) \right\} du_1 \ldots du_n , \]  
(II, 17)
\[ w_0 = \varphi.w , \]  
(II, 18)
\[ u_1 = G(x_1, \ldots, x_n), \]
\[ u_2 = x_2, \]
\[ \vdots \]
\[ u_n = x_n, \]

and

\[
D \left( \frac{x}{u} \right) = \left[ D \left( \frac{u}{x} \right)^{-1} \right]^{-1} = \frac{1}{\partial G/\partial x_1},
\]

with

\[
\frac{\partial G}{\partial x_1} > 0.
\]

Otherwise, from [10], p. 211, form. (8),

\[
\delta^{(k)}(G(x), \varphi) = (-1)^{k} \int_{G} f_{u_1}^{(k)}(0, u_2, \ldots, u_n) du_2, \ldots, du_n,
\]

where

\[
f(u_1, u_2, \ldots, u_n) = \varphi_1(u_1, \ldots, u_n) D \left( \frac{x}{u} \right),
\]

\[
\varphi_1(u_1, u_2, \ldots, u_n) = \varphi(x_1, x_2, \ldots, x_n),
\]

and \( D(\frac{x}{u}) \) is defined by (II,20).

**III. The multiplicative product of the Hankel transform of the \( k \)-th derivative of the Dirac delta in \( u(x_1, x_2, \ldots, x_n) \).**

In this paragraph we shall obtain the following formula

\[
\mathcal{H} \left\{ \delta^{(k)}(u(x)) \right\} \mathcal{H} \left\{ \delta^{(\ell)}(u(x)) \right\} = D \mathcal{H} \left\{ \delta^{(k+\ell+n-2)}(u(x)) \right\},
\]

where \( D \) is a constant given by (III,8).

Taking into account the formula (26), p. 4 of [8], we know that

\[
\mathcal{H} \left\{ \delta^{(k)}(u(x)) \right\} = \mathcal{H} \left\{ \delta^{(k)}(u(x_1, x_2, \ldots, x_n)) \right\} = B_{k,n}(u(y_1, y_2, \ldots, y_n))^{\frac{n-2}{2}+k},
\]

(III,2)
where $\delta^{(k)}(u(x_1, \ldots, x_n))$ is given by the formula (II,22) and

$$B_{k,n} = \frac{1}{2^{2k+n} \frac{n}{2} \Gamma \left(\frac{n}{2} + k\right)} ,$$  \hspace{1cm} (III, 3)

$u(y_1, y_2, \ldots, y_n) \in C^\infty(\mathbb{R}^n)$ be such that the $(n-1)$-dimensional manifold $u(y_1, \ldots, y_n) = 0$ has no critical points.

Then, we have

$$\mathcal{H} \left\{ \delta^{(k)}(u(x)) \right\} . \mathcal{H} \left\{ \delta^{(\ell)}(u(x)) \right\} = B_{k,n}(u(y))^{\frac{n-2}{2}+k} B_{\ell,n}(u(y))^{\frac{n-2}{2}+\ell}$$

$$= B_{k,n} B_{\ell,n}(u(y))^{\frac{n-2}{2}+k+\frac{n-2}{2}+\ell} .$$ \hspace{1cm} (III, 4)

The passage from the second to the third equality of (III,4) is licit. Indeed, we can multiplicate $[u(y)]^\lambda [u(y)]^\mu$ first, as locally integrable functions for $\text{Re} \lambda > 0$, $\text{Re} \mu > 0$, and then, by analytical continuation, for every $\lambda, \mu \in \mathbb{C}$.

Otherwise, we know (cfr. form. (III,2)) that

$$\mathcal{H} \left\{ \delta^{(k+\ell+\frac{n-2}{2})}(u(x)) \right\} = B_{k+\ell+\frac{n-2}{2},n}(u(y))^{\frac{n-2}{2}+k+\frac{n-2}{2}+\ell} .$$ \hspace{1cm} (III, 5)

From (III,4) and (III,5), we have

$$\mathcal{H} \left\{ \delta^{(k+\ell+\frac{n-2}{2})}(u(x)) \right\} = \frac{B_{k+\ell+\frac{n-2}{2},n}}{B_{k,n} B_{\ell,n}} \mathcal{H} \left\{ \delta^{(k)}(u(x)) \right\} . \mathcal{H} \left\{ \delta^{(\ell)}(u(x)) \right\} . \hspace{1cm} (III, 6)$$

That is,

$$\mathcal{H} \left\{ \delta^{(k+\ell+\frac{n-2}{2})}(u(x)) \right\} = D \mathcal{H} \left\{ \delta^{(k)}(u(x)) \right\} . \mathcal{H} \left\{ \delta^{(\ell)}(u(x)) \right\} ,$$ \hspace{1cm} (III, 7)

here

$$D = \frac{B_{k+\ell+\frac{n-2}{2},n}}{B_{k,n} B_{\ell,n}} ,$$ \hspace{1cm} (III, 8)

or, equivalently,

$$D = \frac{\Gamma \left(\frac{n}{2} + k\right) \Gamma \left(\frac{n}{2} + \ell\right)}{2^{2-2} \Gamma \left(n + k + \ell - 1\right)} .$$ \hspace{1cm} (III, 9)

Otherwise, by remembering the Pochhammer symbol, defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)} ,$$ \hspace{1cm} (III, 10)

where $a$ is an arbitrary complex and

$$(a)_0 = 1 ,$$

$$(a)_n = a(a+1) \cdots (a+n) ,$$ \hspace{1cm} (III, 11)

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\[ n = 1, 2, \ldots; \text{we can express the constant } D \text{ (form. (III,9)) in the equivalently form} \]

\[ D = \frac{\Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} \right) \ell}{2^{\frac{n}{2} - 2} (k + \ell - 1) \Gamma(k + \ell - 1)}. \]  

(III,12)

Finally, we note that, taking into account the theorem of identity for Hankel transforms and the formula (III,2), the following formula is valid:

\[ \delta^{(k)}[u(x_1, x_2, \ldots, x_n)] = B_{k,n} \mathcal{H} \left\{ [u(x_1, x_2, \ldots, x_n)]^{\frac{n-2}{2} + k} \right\}, \]  

(III,13)

where \( B_{k,n} \) is the constant given by (III,3).

**IV. The multiplicative product of the Hankel transform of the \( k \)-th derivative of the Dirac delta in \( P(x) \).**

In this paragraph we obtain the multiplicative product of the Hankel transform of the \( k \)-th derivative of the Dirac delta in \( P(x) \).

\[ \mathcal{H} \left\{ \delta^{(k)}(P) \right\} \cdot \mathcal{H} \left\{ \delta^{(\ell)}(P) \right\} = C \mathcal{H} \left\{ \delta^{(k+\ell+\frac{n-2}{2})}(P) \right\}, \]

(IV,1)

where \( C \) is the constant given by (IV,9).

We know (cfr. [4], form. (36), p. 279) that

\[ \mathcal{H} \left\{ \delta^{(k)}(P) \right\} = \frac{1}{2^{2k+\frac{n}{2}} \Gamma \left( \frac{n}{2} + k \right)} Q^{\frac{n-2}{2} + k}, \]

(IV,2)

here

\[ Q = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2, \]

(IV,3)

where \( p + q = n \).

Also, we note by

\[ Q_+ = \begin{cases} Q & \text{if } Q > 0, \\ 0 & \text{if } Q \leq 0. \end{cases} \]

(IV,4)

Therefore, we obtain the following equation

\[ \mathcal{H} \left\{ \delta^{(k)}(P) \right\} \cdot \mathcal{H} \left\{ \delta^{(\ell)}(P) \right\} = \frac{Q_+^{\frac{n-2}{2} + k}}{2^{2k+\frac{n}{2}} \Gamma \left( \frac{n}{2} + k \right)} \cdot \frac{Q_+^{\frac{n-2}{2} + \ell}}{2^{2\ell+\frac{n}{2}} \Gamma \left( \frac{n}{2} + \ell \right)}. \]

(IV,5)
By taking into account the Theorem 2, p. 23 of [5], we can write, equivalently, the formula (IV,5) as

\[ \mathcal{H}\{\delta^{(k)}(P)\}\mathcal{H}\{\delta^{(\ell)}(P)\} = \frac{Q_{+}^{n-2+k+\frac{n-2}{2}+\ell}}{2^{2k+\frac{n}{2}+2\ell+\frac{n}{2}} \Gamma\left(\frac{n}{2}+k\right) \Gamma\left(\frac{n}{2}+\ell\right)} \cdot \] (IV,6)

Otherwise, we have

\[ \mathcal{H}\{\delta^{(k+\ell+\frac{n-2}{2})}(P)\} = \frac{Q_{+}^{n-2+k+\ell+\frac{n-2}{2}}}{2^{2(k+\ell+\frac{n-2}{2})+\frac{n}{2}} \Gamma\left(\frac{n}{2}+k+\ell+(n-2)\right)} \cdot \] (IV,7)

From (IV,6) and (IV,7), we arrive at the following formula

\[ \mathcal{H}\{\delta^{(k)}(P)\}\mathcal{H}\{\delta^{(\ell)}(P)\} = C\mathcal{H}\{\delta^{(k+\ell+\frac{n-2}{2})}(P)\} \cdot \] (IV,8)

where \( C \) is the constant given by

\[ C = \frac{2^{\frac{n}{2}-2} \Gamma(n+k+\ell-1)}{\Gamma\left(\frac{n}{2}+k\right) \Gamma\left(\frac{n}{2}+\ell\right)} \cdot \] (IV,9)

V. The multiplicative product of the Hankel transform of the \( k \)-th derivative of the Dirac delta of \(|x|^2\).

We know that \( P = P(x) = |x|^2 \) if \( p = n \) and \( q = 0 \) in the formula (II,1), so the formula (IV,8) arrives at

\[ \mathcal{H}\{\delta^{(k)}(|x|^2)\}\mathcal{H}\{\delta^{(\ell)}(|x|^2)\} = C\mathcal{H}\{\delta^{(k+\ell+\frac{n-2}{2})}(|x|^2)\} \cdot \] (V,1)

where \( C \) is the constant given by (IV,9).

Bibliography.


[8] M. Aguirre Téllez. The Hankel Transform of the \( k \)-th derivative of Dirac delta in \( u(x_1, x_2, \ldots, x_n) \). To appear.
